Puzzle Compendium

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Abstract

I have always harbored a long-standing fascination with brainteasers. However, after one is solved, it becomes trivial (modulo forgetting the solution). As such, over time, it becomes necessary to find increasingly more difficult puzzles. This is my effort to consolidate those I have encountered which I consider to be some of the more challenging, high-quality, mathematically interesting, or otherwise noteworthy of the bunch.

1 Introduction

What makes a good puzzle? Solvability alone is not enough –"what is 2×3?" poses little challenge and thus little payoff. Difficulty alone is not enough – it may be hard to factorize large semiprimes, but this is also not a particularly interesting task (cryptographers may disagree). In fact, I find that a key theme from cryptography is a clean and concise way to express quality: A good puzzle is a problem which has a solution which is difficult to find, but makes sense (maybe with some hard thinking). The ratio of of difficulty to solve to cleverness of the actual answer is perhaps my best proxy for puzzle quality. The sense of insight one feels from truly understanding the solution to a puzzle is, in my opinion, the most significant indicator of puzzle quality. But this also requires some level of cleverness – a solution that is completely unintuitive but logically sound. For some puzzles, I find that discovering the solution does not provide same the level of satisfaction as understanding how it works. I attempt to highlight some puzzles which I have felt most satisfied this latter feeling. The puzzles are collected thematically, and in some rough semblance of ascending difficulty within sections (though this is not a partial order, and my sense will not necessarily match yours). I make no claim to be the author of any of these puzzles, many of which are folklore. Unless explicitly stated otherwise, you may assume that any of the characters in the following puzzles are perfect logicians.

2 Hat Puzzles

These are puzzles often phrased to involve a sadistic prison warden lining up some number of prisoners and placing hats of various colors on their heads. The prisoners are instructed to each guess the color of their own hat(s), with an offer of freedom contingent upon some rate of successful guesses, and sometimes a punishment in the case of failure. However, they may not look at their own hats and are not allowed to communicate with each other. If the prisoners have advance knowledge of the warden's scheme, can they collectively plan ahead and devise a strategy to improve their chance of success?

There are some non-mathematical loopholes to this problem. For example, one could posit the strategy "everyone lines up, and speak your guess in less than 3 seconds if the next person's hat is blue, and more than 3 seconds if it's red." While these types of tactics may be effective in a real-life scenario, allowing them detracts from the puzzle's quality, so these will not be mentioned as "correct" solutions. The only permitted transfer of information is that which is part of the puzzle.

In the interest of somewhat reframing the power dynamics, I will pose these problems as games played between Papa Gnome and a set of his gnome children. This also has the benefit of appealing to gnomes' well-known penchant for both hats and mathematics.

1. Suppose n children play a game with Papa, where he places either a red or blue hat on each child (he has an infinite supply of any color of hat). They can each see every one else's hat, but not their own. The children guess sequentially (so they can hear the previous guesses), and win if every one of them is able to correctly guess their own hat color. What strategy can the children implement to maximize their chance of winning, and what is the probability?

   Solution:

2. This setup is the same as the previous puzzle, but now Papa has k different colors of hats.

   Solution:

3. Papa places either a red or blue hat on each of $2N$ childrens' heads. They must all guess their own colors simultaneously. The children win if at least N of them are correct.

   Solution:

4. One from reddit. Papa Gnome gathers n children and writes a positive whole number on their backs. Each child can see everyone else's numbers, but not their own, and all the numbers are different. Each child has a red hat and a blue hat in front of them. When Papa rings his bell, everyone as to pick a hat and put them on. Then Papa will line them up in ascending order of their numbers. If their hats alternate red-blue-red-blue-$\cdots$ or blue-red-blue-red-$\cdots$, then the children win. Otherwise Papa wins. What strategy should the children use to maximize their chance of winning?

   Solution:

5. Another from reddit. Papa has n colors of hat. Abby Gnome chooses a positive integer a and a hats are stacked on her head. Billy Gnome chooses a positive integer b and b hats

are stacked on his head. They can both see all of the other child's hats, but none of their own. They will each guess the color of each one of their hats (they cannot hear each other's guesses). The children win if at least one of the $a + b$ guesses is correct. Do they have a winning strategy?

Solution:

In all further hat puzzles, at least one set involved may be infinite. We lay some ground rules for reasoning about the infinitary: You are allowed to freely use the axiom of choice (this is also a hint! More specifically, Hint:). If you do not know what this is, you may find this next stretch of puzzles particularly challenging. You can assume that all actions happen instantaneously, so that gnomes are capable of performing infinitely many actions. Additionally, they have infinite memory (to help take advantage of any results of choice).

6. Papa plays the same game with countably infinitely many children, who each get either a red or blue hat (again, assume he has enough hats of each color for every child). The children then all guess their own hat colors simultaneously. The children win if all but finitely many of them guess correctly.

   Solution:

7. The same setup as the previous puzzle with a countably infinite number of children, but now the children answer in sequence. They win if all but one of them guess correctly.

   Solution:

8. Papa Gnome has a countably infinite number of boxes, each containing a slip of paper with a real number written on it (they do not have to be distinct). He challenges Abby Gnome to the following game: She can look at as many of the numbers as she wants (potentially infinitely many) as long as she leaves at least one box untouched. She then has to guess the value of a number in a box that she has not opened. Prove that Abby can do this with probability $1 - \varepsilon$ for any $\varepsilon > 0$.

Solution:

9. These next few, including solutions, are straight from reddit. The setup is as follows: Papa chooses a (possibly infinite) group of children and a (possibly infinite) pallet of hat colors, which are known to everybody. Colored hats get distributed among the children, with each color potentially appearing any number of times. Each child can see everyone else's hat but not their own. Everyone must simultaneously make a guess about the color of their own hat. Can you find a strategy that ensures:

   (a) If just one child guesses their hat color correctly, then everyone will guess correctly.

   (b) Only finitely many children guess incorrectly.

   (c) Exactly one person guesses correctly, given that the cardinality of people is the same as the cardinality of possible hat colors.

Solution:

10. Papa and three children play a game. He places either a red hat or blue hat on each child's head, uniformly at random. The children have to guess the color of their own hat simultaneously. However, they are also allowed to say "I don't know." The children win if at least one guesses their own hat color correctly, and nobody gets it wrong. What strategy can they use to maximize their chance of winning?

    Solution:

11. This, and the following two, from reddit. Papa plays another game with Abby and Billy. He places an infinite stack of red and blue hats on each of their heads. Abby and Billy must simultaneously guess the sequences of their own hat colors. They win if at least one of them gets infinitely many guesses correct.

    Solution:

12. The same game as above, but with red, blue, and yellow hats.

    Solution:

13. The same game as above, but instead of colors, each child has an infinite stack of real numbers on their head. You may assume the Continuum Hypothesis.

    Solution:

For further mathematical reading concerning hat problems, see here, here, and here.

3 Probability Puzzles

This section contains riddles related to elementary probability. The machinery required is hopefully not too complicated (since I don't know much probability).

14. Pick two real numbers x, y uniformly at random from $[0, 1]$. For what value of z is the probability that $|x - y| < z$ equal to the probability that $|x - y| > z$?

    Solution:

15. Alice picks a number $\alpha \sim U([0, 1])$. Bob repeatedly samples $\beta_i \sim U([0, 1])$ until he gets a number greater than α. What is the expected number of samples for Bob?

    Solution:

16. Given a biased coin with probability p of heads, how can you simulate a fair coin using only a finite number of flips in expectation?

    Solution:

17. Given a fair coin, how can you simulate a biased coin with probability p of heads using only a finite number of flips in expectation?

    Solution:

18. You have a large pot of spaghetti, with N strands. They are all tangled up, but the ends of each strand are both sticking out of the pot. However, you can't tell which ends are which. You grab two ends at random and tie them together, repeating this until no ends are free. If you untangle the spaghetti, there will be some finite number of loops formed. What is the expected number of loops?

    Solution:

4 Graph Theory Puzzles

This section contains riddles related to finite graph theory. Depending on your background, these may not be too difficult. Some graph conventions: $G = (V, E)$ for both directed and undirected graphs, and we typically use n to denote the number of vertices and m the number of edges. The neighbors $N(v)$ of a vertex v are v and all adjacent vertices.

19. Let G be a graph on 3^n vertices. For each vertex v, count the number of vertices which are not adjacent to v, $|V \setminus N(v)|$. If the sum of these counts over all vertices is less than 3^k for some integer k, prove that there is a vertex with degree at least $3^n - 3^{k-n}$.

    Solution:

20. Suppose that G is a graph with chromatic number 6. Prove that there are two vertex-disjoint odd cycles in G.

Solution:

21. Let G be a directed tournament graph (a directed graph with exactly one edge between every pair of distinct vertices). We say a vertex u *dominates* another v if either: (1) $(u, v) \in E$, or (2) there exists a vertex w such that $(u, w), (w, v) \in E$. Prove that there is a vertex that dominates every other vertex.

Solution:

22. Recall that $\chi(G)$ is the minimum number of colors needed to color an undirected graph G. G is called *color-critical* if $\chi(G - v) < \chi(G)$, for all $v \in V$. If $G = (V, E)$ is color-critical then show that $\deg(v) \geq \chi(G) - 1$, for all $v \in V$.

Solution:

23. Let $k \in \mathbb{Z}^+$. Show that, for any graph G, it is always possible to k-color the vertices such that at least $\frac{k-1}{k}|E|$ edges have different-colored endpoints. (Equivalently, any graph contains a k-partite subgraph with at least $\frac{k-1}{k}|E|$ edges).

Solution:

24. Let $k \geq 3$. Suppose that G satisfies the following property: Every set of k vertices in G shares exactly one common neighbor (vertices are not considered their own neighbors). Find an upper bound on the max degree in G.

Solution:

25. Let $k \in \mathbb{Z}^+$ and let G be a graph where, for every vertex v, $\deg(v) = \frac{1}{k} \sum_{u \in N(v)} \deg(u)$. Prove that the maximum possible degree of any vertex in such a graph is $k^2 - k + 1$ and show that this bound is tight.

Hint:

Solution:

5 Number Theory Puzzes

While I say "Number Theory," there's no real advanced machinery here. These are really puzzles about the integers, and mostly about divisibility (though the solutions largely draw from other techniques).

26. Prove that in any set of n integers, there is either a number divisible by n, or there are two numbers whose difference is divisible by n.

Solution:

27. Prove that in any set of $n + 2$ integers, there are two numbers such that either their sum or difference is divisible by $2n$. (I think this is an old Putnam question but haven't been able to track it down)

Solution:

28. Let p be a three-digit prime. Prove that there exists repdigit (a number which consists only one digit repeated) which is divisible by p.

    Solution:

29. Prove, for any natural number n, that it is possible to select 2^n numbers from any arbitrary collection of 2^{n+1} integers (not necessarily distinct) such that that sum of the 2^n numbers is divisible by 2^n.

    Solution:

6 Math Puzzles

This section consists of a collection of general mathematical riddles. The math required varies from pre-high school to graduate-level, though actual solutions may be trickier than this suggests.

30. From the wu forums. The sum of N real numbers is 20. If the sum of the smallest three is 5, and the sum of the largest three is 7, what are the possible values of N?

    Solution:

31. Alice and Bob play the following game. There are $2N$ coins of various denominations in a row on a table. On each player's turn, starting with Alice, they must take either the leftmost or rightmost remaining coin. Prove that Alice has a strategy that guarantees she wins at least as much total money as Bob.

    Solution:

32. Prove that every real number can be written as a finite sum of real numbers, each of which contains only the digits 0 and 3.

    Solution:

33. Is it possible to pick 7 points in $\mathbb{R}^2$ such that among any set of three, there are two points exactly distance 1 away from each other?

    Solution:

34. It is well-known that the derivative of x^2 is $2x$. But if we rewrite the product $x \cdot x$ as a sum, the following happens:

$$\begin{aligned}\frac{d}{dx}(x^2) &= \frac{d}{dx}(x + x + \cdots + x, x \text{ times}) \\ &= \frac{d}{dx}(x) + \frac{d}{dx}(x) + \cdots + \frac{d}{dx}(x), x \text{ times} \\ &= 1 + 1 + \cdots + 1, x \text{ times} \\ &= x\end{aligned}$$

    What went wrong?

Solution:

35. Alice and Bob play the following game: Alice chooses a polynomial f with positive integer coefficients. Bob can adaptively query the value of f on some points $x \in \mathbb{Z}$. After k guesses, he must guess the identity of f and wins if he is correct; otherwise, Alice wins. What is the minimum k for which Bob has a winning strategy?

Solution:

36. The same game as above, but now Bob is allowed to query the value of f on $x \in \mathbb{R}$.

Solution:

37. Suppose you have a sequence of $ab + 1$ distinct integers. Prove that there exists a strictly increasing sequence of length $a + 1$ or a strictly decreasing sequence of length $b + 1$.

Solution:

38. Suppose you are given a k-coloring of the d-dimensional integer lattice $\mathbb{Z}^d$. Prove that there exists a d-hyperrectangle such that all of its vertices are the same color. (This is a generalization of an old USAMTS problem)

Solution:

39. From the Williams Conundrum. As Thanksgiving is rapidly approaching, many turkeys are understandably worried. Several of them have gotten together and convinced humanity to accept the following challenge (rather than settling things with the sword).

The turkeys will create a polynomial $P(x)$ such that, no matter what integer k the humans give them, the output $P(k)$ will be an integer. If they can do this with one of the coefficients of $P(x)$ being $1/2013$, then no turkeys will be eaten for the rest of 2013.

Can the turkeys succeed? More generally, if you give them finitely many years (say $n+1$ years) can they create a similar polynomial which has each of $1/2013, 1/2014, \dots, 1/(2013+n)$ as coefficients?

Solution:

40. The coast guard is trying to track down a rogue pirate ship somewhere in $\mathbb{R}^2$. The only thing they know is the last time the ship was seen, and at that time, it was at an unknown lattice point (a element of $\mathbb{Z}^2$) and has set a course to travel along a straight line with that passes through at least one other lattice point at an integer speed (in units/day). The coast guard has one plane they can send to check a single location (anywhere they want) every day, with a vision radius of 1. Can they devise a strategy to guarantee that they find the ship in a finite amount of time?

Solution:

41. Another from reddit. Alice and Bob play a game on the reals. Alice starts by selecting an uncountable subset $S_0 \subseteq \mathbb{R}$. Then they alternate selecting $S_1, S_2, \ldots$, each of which is uncountable, such that $S_i \supseteq S_{i+1}$. They play for a countably infinite number of steps. Alice wins if $\bigcap_{i\in\mathbb{N}} S_i$ is empty; otherwise Bob wins. Who has a winning strategy?

Solution:

7 Other Puzzles

Though mathematical knowledge may be helpful, it is less explicitly required for this more general selection. These rely primarily on logical thinking.

42. A wealthy man dies and leaves, among his other assets, his 17 horses to his children. They are to be divided in the following ratio: The eldest child receives 1/2, the middle child receives 1/3, and the youngest receives 1/9. How are the children to fairly divide the horses if none of them feel comfortable with the idea of keeping a fractional horse?

Solution:

43. You are placed in a dark room with two identical tables. One table contains 100 coins of various denominations and the other is empty. You are told that exactly 10 of the coins are heads-up, but it is too dark to see them. Your task is to move some of the coins from the full table to the empty one so that at the end, all coins are on one of the two tables and both tables have the same number of coins with heads up. You are forced to wear gloves, so you cannot feel the surfaces of the coins at all. What is your best strategy?

Hint:

Solution:

44. Suppose you have 25 horses. A faster horse will always beat a slower horse, but horses don't always run at the same speed. If you can race 5 horses at a time, how many races are needed to determine the fastest 3 horses?

Solution:

45. There are two doors, with a prize behind one and nothing behind the other. Each door is guarded a knight or a knave (knights always tell the truth and knaves always lie) – but it's possible that they're both knights or both knaves. Can you ask one yes-no question to determine which door hides the treasure?

Hint:

Solution:

46. Consider an arbitrary triangle T. Show that for any set of 7 points in the interior of T, it is always possible to pick three points such that the area of the triangle formed by these points is at most $\frac{1}{4}$ the area of T.

Solution:

47. From Steven Miller's puzzles. I am an honest person, and am thinking of one of three numbers: 1, 2 or 3. You may ask me EXACTLY one yes-no question, I will answer truthfully. What question should you ask?

    Hint:

    Solution:

48. Alice and Bob play a game on a large circular table. Starting with Alice, they take turns placing identical coins on the table. The coins must be placed flat, cannot move previously placed coins, cannot overlap other coins, and must lie entirely on the table. The winner is the last player who is able to place a coin. Who has a winning strategy?

    Solution:

49. On a certain island live 100 villagers, 50 with brown eyes and 50 with blue eyes. The village has a peculiar rule: anyone who knows that they have blue eyes must leave the island at sunset. Luckily, there are no mirrors in the village, and nobody knows for certain the color of their own eyes. There's also no restrictions on sight – nobody is blind, and everybody can see everyone else's eyes. One day, a shipwrecked sailor stumbles upon the village and is nursed back to health. Before he departs, the village holds a festival in his honor, where he announces to the crowd that "at least one villager has blue eyes." Now, this is information that all the villagers already know – they can see that there are blue-eyed villagers among them. What effect, if any, does this announcement have?

    Hint:

    Solution:

50. There is a magic trick performed by two magicians, Alice and Bob, with one regular, shuffled deck of 52 cards (assume that the cards are rotationally symmetric). Alice asks Carol to randomly select 5 cards out of a deck and hand the 5 cards back to Alice. After looking at the 5 cards, Alice picks one of the 5 cards and gives it back to Carol. Alice then arranges the other four cards in some way, and gives those 4 cards face down, in a neat pile, to Bob. Bob

looks at these 4 cards and then announces the card that Carol is holding. How does this trick work?

Solution:

51. Papa Gnome plays a game with N children. He has a secret room with no windows containing a single box with two switches which can each be in either "up" or "down" positions – the initial state of the switches is unknown. Every day, he brings one child into the room, where they must flip exactly one of the switches and then leave – they can't use the room to send any sort of other information, and they aren't allowed to discuss anything once the game starts. He doesn't have to bring the children in any particular order, although there is the guarantee that at any point in time, for each child, there will be some future time at which that child will be brought to the room. At any point in time, any child can declare that everyone has been into the room at least once. If they are correct, the children win; otherwise, they lose. Do they have a winning strategy?

Hint:

Solution: